Scattering by Finite Periodic P T -Symmetric Structures

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In this work, we study the transmission properties of one dimensional finite periodic systems with $\mathcal{PT}$-symmetry. A simple closed form expression is obtained for the total transmittance from a lattice of N cells, that allows us to describe the transmission minima (maxima) when the system is in the $\mathcal{PT}$-unbroken (broken) phase. Utilizing this expression, we provide the necessary conditions, *independent* of the number of cells, for the occurrence of a CPA-laser for any finite $\mathcal{PT}$-symmetric periodic potential. Under these conditions, we provide a recipe for building finite periodic structures with near perfect absorption and extremely large amplification.

Studying the interplay between losses and gain in wave propagation has recently attracted considerable attention, stimulated by the discovery of $\mathcal{PT}$-symmetric [1] systems. Theoretical efforts where initially focused on extending Hermitian quantum theories [2, 3] using non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians with real spectra. It has been however realized that the notion of $\mathcal{PT}$-symmetry and the corresponding phenomena can be readily extended to other physical systems. Experiments emulating $\mathcal{PT}$-symmetric Hamiltonians are now performed in diverse physical settings including optical waveguides [4, 5], micro-ring resonators [6–10], audible acoustics [11, 12], optomechanics [13], spin-waves [14], or atomic systems [15] The interest in such systems is motivated by the extraordinary wave properties, especially around exceptional points [16, 17] which are otherwise unattainable in Hermitian systems including, unidirectional propagation [18–20], enhanced sensitivity [21] and coherent perfect absorbers-lasers (CPA-lasers) [22, 23]. The CPA-laser, especially in optics, is of great technological interest since it may lead to devices which are multifunctional, acting on the same time as absorbers, lasers or modulators. Experimental observation of the CPA-laser was first realized in an electronic circuit analogue [24] and it has been very recently reported in an optical setting [25].

$\mathcal{PT}$-symmetry requires a delicate exact balance between gain and loss. which makes the experimental observation of the PT-phase transition as well as the CPA-laser a very challenging task. One way to bypass the difficulties is to try and build elemental two-component systems and fine-tune loss/gain the obtain exact $\mathcal{PT}$-symmetry. On the other hand, periodic structures with $\mathcal{PT}$-symmetry are of special interest [26–30], since they have been shown to have interesting properties including unusual band structure and Bloch oscillations [31–35]. They offer a unique opportunity for generating novel devices, and constitute a promising setting to overcome the consequences of losses in many applications including the growing field of metamaterials.

Although many studies exist for the case of infinite periodic $\mathcal{PT}$-symmetric systems, the case of scattering in a finite periodic systems composed of N number of cells has been less investigated. In many cases (see for example Ref. [18] and the experimental works [25, 37, 38]) the finite system is studied using a coupled mode theory around the Bragg points and an approximate transfer matrix describes the relevant phenomena.

It was recently shown in Ref. [39] that asymmetric transmission resonances (ATR) [40] (exceptional points of the scattering matrix) and the CPA-laser points of a finite number $\mathcal{PT}$-symmetric dielectric layers highly depend on the number of cells. A finite lattice of N-quantum $\mathcal{PT}$-symmetric scatterers was studied in Ref. [41] focusing on the difference between parallel and in-series coupling of the scatterers, but also relating the singular points of the scattering matrix the unit cell and the N-cell structure. Even more recently the singular value spectrum of a finite periodic electromagnetic structure was studied in Ref. [42], focusing on the CPA-laser point.

The purpose of this article is to give simple closed form expressions, describing the transmission and the CPA-laser points, from an arbitrary one dimensional finite periodic $\mathcal{PT}$-symmetric scatterer. The expression for the total transmission from the finite system is given in Eq. (7), which depends on the unit cell transmission, the Bloch phase and the total number of cells. From this expression we can deduce the number of transmission resonances, the ATRs and the CPA-laser points. Furthermore we obtain an envelope function, Eq. (9), which captures the minima of transmission in the $\mathcal{PT}$-unbroken phase or the maxima in the $\mathcal{PT}$- broken phase. Using this simple form we obtain the necessary conditions for a CPA-laser, in any finite periodic $\mathcal{PT}$-symmetric potential in 1D, independently of the number of cells (length). These conditions are given in Eq. (10). The exact CPA-laser which depends on the number of cells N is also found here analytically. Finally we show that when the necessary requirements obtained by the envelope function are met, near perfect absorption and extremely strong amplification at the same frequency can be obtained even away from the CPA-laser.

In the following, we study one dimensional scattering systems satisfying the stationary Schrödinger equation

$$\psi'' + (k^2 - V(x))\psi = 0, \tag{1}$$

or the Helmholtz equation relevant to optical Bragg gratings

$$\psi'' + k^2 n^2(x)\psi = 0 \tag{2}$$

where $\psi(x)$ is the wave field and primes denote derivatives with respect to x. The $\mathcal{PT}$-symmetry of the potential is established when $V(x) = V^*(-x)$, where star denotes complex conjugation, while the normalized refractive index $n(x)$ also is $\mathcal{PT}$-symmetric when $n(x) = n^*(-x)$. We study space periodic potentials of period l satisfying $V(x+l) = V(x)$ and $n(x+l) = n(x)$ as shown in Fig. 1 (a). For both equations, the scattering matrix for the unit cell with length l has the form

$$\mathbf{S_1} = \begin{pmatrix} r_L & t \\ t & r_R \end{pmatrix}, \tag{3}$$

where t, r_L and r_R are the transmission, reflection from left and right coefficients respectively. The transfer matrix of the unit cell defined in the region $-l/2 \leq x \leq l/2$, is $\mathcal{PT}$-symmetric and can be written as:

$$\mathbf{M_1} = \begin{pmatrix} 1/t^* & r_R/t \\ -r_L/t & 1/t \end{pmatrix}. \tag{4}$$

$\mathbf{M_1}$ has a unitary determinant which leads to the following relation $r_L r_R = t^2(1 - T_1^{-1})$ and subsequently to the following "conservation law" $|T_1 - 1| = \sqrt{R_L^{(1)} R_R^{(1)}}$. Here $T_1 = |t|^2$ is the total transmittance from the unit cell and $R_{L,R}^{(1)} = |r_{L,R}|^2$ the total reflectances of the unit cell from left and right. Since the system does not conserve time reversal symmetry in general $R_L^{(1)} \neq R_R^{(1)}$.

Depending on the parameter values of the unit cell scatterer, the transmittance may either be $T_1 < 1$ or $T_1 > 1$. Besides, the eigenvalues of $\mathbf{M}_1$ can be written as $\lambda_{1,2} = \exp[i\phi]$ resulting in

$$\cos\phi = Re(1/t). \tag{5}$$

In the case of an infinite periodic potential the phase ϕ corresponds to the Bloch phase.

Transmittance— We now focus on the scattering from a finite periodic structure composed by N cells. Thus, both the potential $V(x)$ or the refractive index $n(x)$ are $\mathcal{PT}$-symmetric in a region $-L/2 < x < L/2$ where the total length of the scatterer is $L = Nl$. Using the classical Chebyshev identity we can write the transfer matrix for a periodic potential composed by N cells [39, 42], which has the form

$$\mathbf{M}_N = \begin{pmatrix} \frac{1}{t^*}\frac{\sin(N\phi)}{\sin\phi} - \frac{\sin(N-1\phi)}{\sin\phi} & \frac{r_R}{t}\frac{\sin(N\phi)}{\sin\phi} \\ -\frac{r_L}{t}\frac{\sin(N\phi)}{\sin\phi} & \frac{1}{t}\frac{\sin(N\phi)}{\sin\phi} - \frac{\sin(N-1\phi)}{\sin\phi} \end{pmatrix} \tag{6}$$

Since the potential is still $\mathcal{PT}$-symmetric, the conservation relation is now generalized to $|T_N - 1| = \sqrt{R_L^{(N)} R_R^{(N)}}$. Using the fact that $\mathbf{M}_N$ also has a unitary determinant, and that it is also $\mathcal{PT}$-symmetric, from

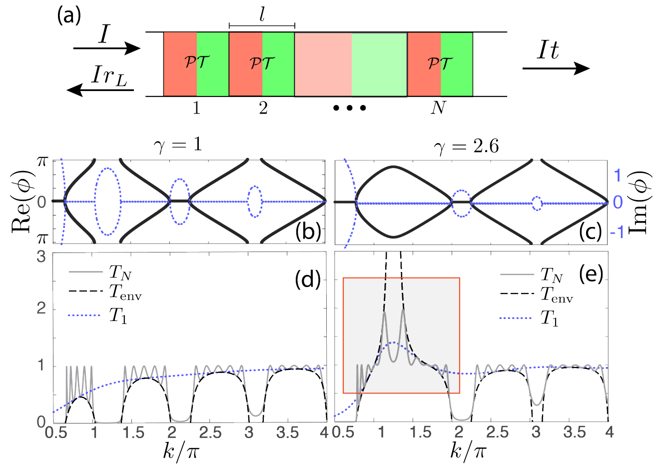

FIG. 1: Panel (a): schematic of the unit cell and the finite periodic potential of N cells. Arrows indicate an incoming wave I form the left and the corresponding transmitted and reflected waves. Panels (b) and (c): the real and imaginary part of the Bloch phase ϕ as a function of wavenumber k/π obtained for the unit cell of Eq. (8). Panels (e) and (d) the transmittance T_N through a finite lattice of $N = 6$ cells as a function of k/π. Left and right columns correspond to gain/loss parameter $\gamma = 1$ and $\gamma = 2.6$ respectively, while in all cases $\lambda = 6$ and $l = 1$.

Eq. (6) we obtain the total transmission from N cells as

$$\frac{1}{T_N} = 1 + (\frac{1}{T_1} - 1)\frac{\sin^2(N\phi)}{\sin^2\phi}. \tag{7}$$

The transmission is obtained as a function of the phase ϕ, the transmission of the unit cell T_1 and the number of cells N. It is very interesting to note that Eq. (7) has the same form as the transmission through a periodic scatterer without loss or gain [43]. In Ref. [43] the expression of Eq. (7) is derived for the conservative system exploiting the fact that the scattering matrix $\mathbf{S}_1$ is unitary. On the contrary, for the $\mathcal{PT}$-symmetric case we use the "generalized unitarity relation" (see for example Ref. [40]). To our knowledge this simple and useful expression has not been established in the literature for $\mathcal{PT}$-symmetric periodic potentials.

The simplicity of Eq. (7) allows us to identify most of the characteristics of the scattering problem with simple arguments, and as we present below, it also permits to extract the necessary conditions for CPA-laser in any 1D finite periodic system. To illustrate the use of Eq. (7) we study a $\mathcal{PT}$-symmetric periodic potential composed by δ barriers for Eq. (1). The unit cell is described by the following potential

$$V(x) = \lambda\delta(x) - i\gamma\left(\delta(x - \frac{l}{4}) - \delta(x + \frac{l}{4})\right), \tag{8}$$

defined in the region $-l/2 < x < l/2$ with a total length l.

In the unbroken phase ($T_1 < 1$), the finite periodic $\mathcal{PT}$-symmetric has a similar behavior as in the conservative

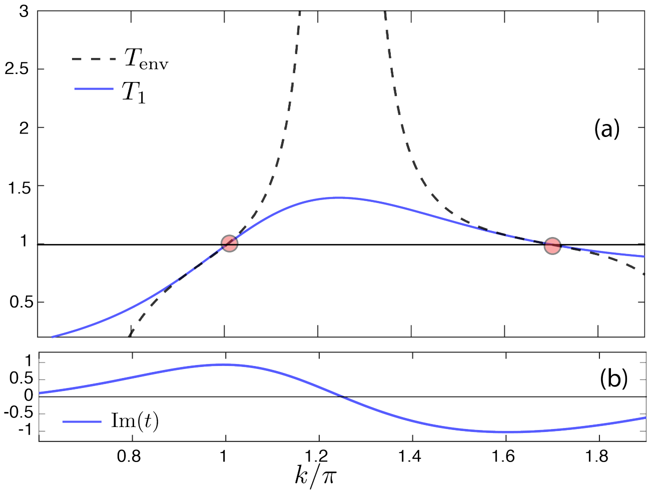

FIG. 2: Panel (a): The transmittance T_1 (dashed) and the envelope function $T_{\rm env}$ (solid) around the $\mathcal{PT}$-broken parametric region where $T_1 > 1$ for $\gamma = 2.6$. Panel (b): The imaginary part of the unit cell transmission coefficient t in the same region.

case (no loss and no gain), and in order to be self contained we review these properties [43]. The transmission T_N, acquires $N-1$ transmission resonances (TR) within each propagating band, corresponding to the values of $N\phi = n\pi$ with $n = 1, 2 \ldots N-1$. An example of the transmittance from $N = 6$ cells of the form of Eq. (8), in the $\mathcal{PT}$-unbroken phase, is shown in Fig. 1 (d), featuring 5 TR in each band. Note that, according to Eq. (7), if the unit cell admits additional resonances ($T_1 = 1$) these have to be added to the $N-1$ stemming from the periodicity. Importantly, the aforementioned reasoning for counting the TRs is based on the fact that the Bloch phase ϕ inside each transmitting band is a *monotonous* function describing the region $\phi \in [0, \pi]$, for the unbroken phase. The Bloch phase of the unit cell (8) with $\gamma = 1$, is shown in Fig. 1 (b). Additionally, the minima of transmission inside each band, appear when $N\phi = n\pi/2$.

In the broken phase ($T_1 > 1$), according to Eq. (7), the transmission changes drastically. Although there exist TRs at $N\phi = n\pi$, now there also peaks of superradiant transmission with $T_N > 1$ at $N\phi = n\pi/2$. Furthermore, for values of k, between the beginning of the first band and $k = 2\pi$ which is now a region of propagation, ϕ is no longer monotonic. In fact, it starts from $\phi = 0$ and returns to 0 without passing from $\phi = \pi$ as shown in Fig. 1 (c). An important consequence of the trajectory of ϕ is that the number of transmission resonances and peaks is now system dependent, and does not follow a prescribed rule as in the conservative system.

Envelope transmission function and CPA-laser condition— A simple envelope function [43], which is *independent* of the number of cells N can be obtained by considering the points where $\sin(N\phi) = 1$ which, using Eq. (7), leads to

$$\frac{1}{T_{\rm env}} = 1 - \frac{(1 - \frac{1}{T_1})}{\sin^2 \phi}. \tag{9}$$

We first note that when unit cell is in the unbroken phase ($T_1 < 1$), the envelope function $T_{\rm env}$ describes the minima of the total transmission T_N [43] in the propagating bands, as is the case for a conservative system. This is shown in Fig. 1 (d) by the dashed black line. Even for a lattice as small as with six cells, this N independent function captures the envelope of the minima efficiently. On the other hand, when the unit cell is in the broken phase ($T_1 > 1$) the function $T_{\rm env}$ now describes the maxima of T_N as shown in Fig. 1 (e). Importantly, as it is shown in Fig. 1 (e), the envelope function $T_{\rm env}$ becomes infinite in the region where $T_1 > 1$ when, according to Eq. (9), $\sin^2 \phi = (1 - 1/T_1)$. Such an infinite transmission in a $\mathcal{PT}$-symmetric potential is known to correspond to a CPA-laser point [22, 23]. The CPA-laser corresponds to the case where one eigenvalue of the total scattering matrix goes to infinity (laser) while the other vanishes (absorber). Due to this property, when the system is found to exhibit huge transmission it also expected to act as near perfect absorber at the same frequency and for the same parameters.

Using Eq. (5) we find the necessary condition, which is N *independent* , in order for a finite periodic system to feature a CPA-laser

$$|t| > 1, \quad \text{and} \;\; \mathrm{Im}(t) = 0. \tag{10}$$

The condition of Eq. (10) depends solely on t, is independent of the number N of scatterers, and is necessary in order to obtain a CPA-laser in the periodic structure. In Fig. 2 (a), we show the single cell transmission T_1 (solid line) corresponding the same values as in Fig. 1 (e), around the superradiant region. With the dashed line, we plot $T_{\rm env}$ which diverges at the value of the point where the imaginary part of t [see Fig. 2 (b)] crosses zero.

For the finite system with N cells, Eq. (10) designates the parametric region where the CPA-laser is able to appear. The exact condition for a CPA-laser requires $N = (2n+1)\pi/2\phi$ with arbitrary n, and if satisfied it will lead to infinite transmittance. In many realistic applications, after a sufficiently high amplification of the wave field nonlinear effects become important and the linear theory is no longer valid. It thus practical, to propose structures able to nearly absorb and efficiently amplify incoming waves but with a finite rate.

Here we like to stress the fact that having a single cell satisfying Eq. 10, allows one to then vary the number of cells and to obtain huge amplification/absorption. For the case of $\gamma = 2.6$ used in the previous examples, we plot in Fig. 3(a) the maximum of transmittance as a function of the number of cells. We observe that the maximum transmission varies significantly even with a small change in the number of cells. The bottom part of the curve in Fig. 3(a), appears to be saturating for large lattices.

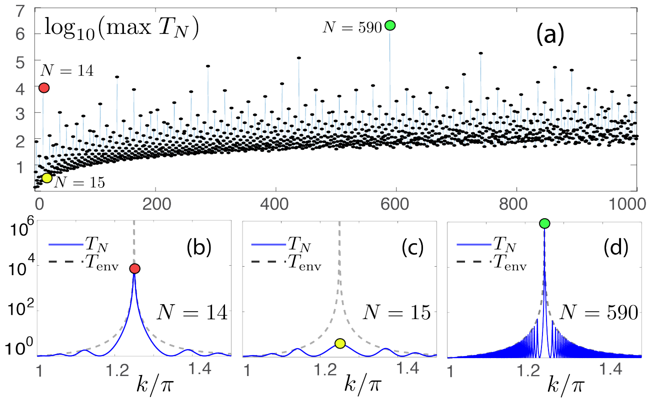

FIG. 3: Panel (a): The maximum of the logarithm of T_N as a function of the total number of cells N. Panels (b), (c) and (d): The transmittance T_N for three different cases as indicated by the number of cells. Parameters used are the same as in the right column of Fig. 1

Importantly, we observe different peaks, within the range of N plotted here, which correspond to an amplification of the wave up to 10^7 times.

Examples of the transmission for three different number of cells are shown in Fig. 3 (b), (c) and (d). In the case of Fig. 3 (b), we have chosen $N = 14$ since it appears to have a huge maximum transmission, in the case of only a few unit cells. In contrast just by adding 2 cells, the transmission in Fig. 3 (c) has a maximum less than 10. Furthermore, using panel (a), we can choose the maximum possible transmission in this range of cells, which appears at $N = 590$, and the corresponding T_N is shown in Fig. 3 (d), reaching upt to an amplification of 10^6.

It is interesting to note that the unit cell crosses from the unbroken to the broken phase through a TR of the unit cell with $T_1 = 1$ which is also a TR of the finite periodic structure. Two such TRs are indicated with (red) circles in Fig. 2 (a). According to the modified conservation law for $\mathcal{PT}$-symmetric systems [see below Eq. (4)], here there is a totally asymmetric reflection since either $R_L^{(1)}$ or $R_R^{(1)}$ is zero (except if there is an accidental zero for both reflections). This transmission resonance was discussed in Ref. [40] and corresponds to an exceptional point of the scattering matrix of the unit cell. Due to the form of the transfer matrix in Eq. (6), the reflections from the finite lattice $R_{L,R}^{(N)}$ are analogous to the ones of the unit cell, and thus this ATRs also appear for the periodic structure.

Electromagnetic gratings— Now, we apply the aforementioned results in a system that has been extensively studied in the context of $\mathcal{PT}$-symmetry and periodicity i.e. a lattice of electromagnetic gratings or slabs. This is one of the few systems which has exhibited a CPA-laser experimentally in optics [15] and it is very promising for for next-generation photonic integrated circuits. For such a system the transverse electric field satisfies Eq. (2) with

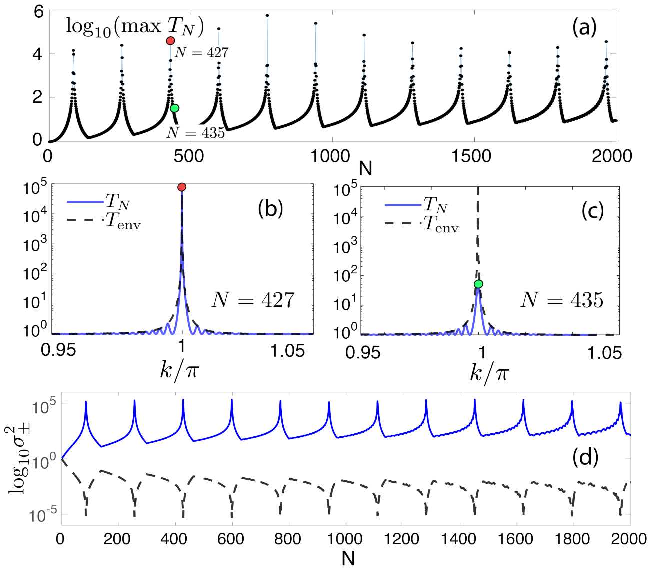

FIG. 4: Panel (a), (b) and (c): same as the corresponding panels of Fig. 3 but for the case of the electromagnetic slabs with parameters $n_1 = 0.02$, and $\gamma = 1.1$. Panel (d) shows the logarithm of the singular values $\sigma_\pm^2$ of the scattering matrix as a function of the number of cells.

a refractive index $n(x)$ given by

$$n(x) = 1 + n_1 \left[\mathrm{sgn}\left(\cos 2lx\right) + i\gamma \,\mathrm{sgn}\left(\sin 2lx\right)\right]. \quad (11)$$

where we consider a piecewise constant refractive index in each slab of length $l/4$. For this particular distribution of the refractive index, it is known [28–30] that the PT-broken phase appears for $\gamma \geqslant 1$. The system is usually analyzed by the use of a coupled mode theory, valid around the Bragg frequency. The length of the total finite lattice, and thus the number of cells, is then chosen using the transfer matrix stemming from the coupled mode theory.

Since for the unbroken phase we have shown that the transmission properties are similar to the conservative case, here we directly focus on the $\mathcal{PT}$-broken phase with $\gamma > 1$. In particular by choosing $n_1 = 0.02$ and $\gamma = 1.1$ we identify the region where the conditions of Eq. (10) are satisfied. In fact the CPA-laser for this settings is found close to the Bragg wavenumber as it is expected [15, 28, 42]. Fig. 4 (a) shows the logarithm of the maximum transmission T_N as a function of the number of cells N. The dependence on the number of cells shows a structured pattern in contrast to the one of Fig. 3. However, also in this case the minimum of $\log(\max T_N)$ appears to saturate to a limiting value for large N. Its is again shown that by satisfying Eq. (10), we can change the number of cells in order to obtain configurations with huge amplification/absorption. Two examples, of the transmission are shown in Fig. 4 (b) and (c), illustrating again how the number of cells can be used in order to obtain large amplification.

In order to describe quantitatively, both amplification and absorption for the case of the electromagnetic slabs, we perform a singular value decomposition of the scattering matrix from N cells. The singular values $\sigma_{\pm}$ are both real and non-negative, and more importantly they are connected through $\sigma_+\sigma_- = 1$ for a 1D $\mathcal{PT}$-symmetric system [42].

For a given incoming wave of the form $\vec{X} = [\psi^+, \psi^-]^T$ where $\psi^{\pm}$ are the forward and backward propagating waves, one can define the output to input ratio as $\Theta = \|\mathbf{S}\vec{X}\|/\|\vec{X}\|$. Then the two singular values correspond to the $\sigma_- = \min[\Theta]$ and $\sigma_+ = \max[\Theta]$, and quantify the ability of the scatterer to absorb or amplify an incoming wave $\vec{X}$. At the CPA-laser point, the singular values become $\sigma_- = 0$ and $\sigma_+ \to \infty$. Here to illustrate the absorptive and amplifying properties of the system Fig. 4 (d), we plot $\sigma_{\pm}$ as a function of N. It is found that by satisfying the conditions of Eq. (10) and varying the number of cells, one is able to obtain a high contrast of absorption and amplification at the same frequency. The dependence on the number of cells is also very sharp in this case.

In summary, we studied finite periodic $\mathcal{PT}$-symmetric scatterers in one dimension. We have obtained closed form expression for the transmission from a finite system of N cells, as a function of the single cell transmission, the Bloch phase and N. A simple envelope function *independent* of the number of cells was obtained, describing the minima (in the $\mathcal{PT}$-unbroken phase) or the maxima (in the $\mathcal{PT}$-unbroken phase) of the total transmission. Using this function we find the necessary conditions for CPA-laser, for an arbitrary 1D periodic 1D system, depending only on the unit cell. Although the exact CPA-laser point depends on its total length, here we show that when the necessary conditions, obtained for the unit cell, are met by varying the number of cells, one can achieve huge amplification and high absorption at the same frequency.

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